Topological-antitopological fusion equations, pluriharmonic maps and special Kähler manifolds

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March 31, 2004

Abstract

We introduce the notion of a tt*-bundle. It provides a simple definition, purely in terms of real differential geometry, for the geometric structures which are solutions of a general version of the equations of topological-antitopological fusion considered by Cecotti-Vafa, Dubrovin and Hertling. Then we give a simple characterization of the tangent bundles of special complex and special Kähler manifolds as particular types of tt*-bundles. We illustrate the relation between metric tt*-bundles of rank r and pluriharmonic maps into the pseudo-Riemannian symmetric space $\mathrm{GL}(r)/\mathrm{O}(p,q)$ in the case of a special Kähler manifold of signature $(p,q)=(2k,2l)$. It is shown that the pluriharmonic map coincides with the dual Gauß map, which is a holomorphic map into the pseudo-Hermitian symmetric space $\mathrm{Sp}(\mathbb{R}^{2n})/\mathrm{U}(k,l)\subset \mathrm{SL}(2n)/\mathrm{SO}(p,q)\subset \mathrm{GL}(2n)/\mathrm{O}(p,q)$, where $n=k+l$.

This work was supported by the 'Schwerpunktprogramm Stringtheorie' of the Deutsche Forschungsgemeinschaft. Research of L. S. was supported by a joint grant of the 'Deutscher Akademischer Austauschdienst' and the CROUS Nancy-Metz.

Contents

1 tt*-equations and pluriharmonic maps

Definition 1 *A* tt*-bundle (E, D, S) *over a complex manifold* (M, J) *is a real vector bundle* $E \to M$ *endowed with a connection* D *and a section* $S \in \Gamma(T^*M \otimes \mathrm{End}\, E)$ *which satisfy the* tt*-equation

$$R^\theta = 0 \quad \textit{for all} \quad \theta \in \mathbb{R}\,, \tag{1.1}$$

where R^θ *is the curvature tensor of the connection* D^θ *defined by*

$$D^\theta_X := D_X + (\cos\theta) S_X + (\sin\theta) S_{JX} \quad \textit{for all} \quad X \in TM\,. \tag{1.2}$$

A metric tt*-bundle (E, D, S, g) *is a tt*-bundle* (E, D, S) *endowed with a possibly indefinite* D*-parallel fibre metric* g *such that for all* $p \in M$

$$g(S_X Y, Z) = g(Y, S_X Z) \quad \textit{for all} \qquad X, Y, Z \in T_pM\,. \tag{1.3}$$

A unimodular metric tt*-bundle (E, D, S, g) *is a metric tt*-bundle* (E, D, S, g) *such that* $\mathrm{tr}\, S_X = 0$ *for all* $X \in TM$*. An* oriented unimodular metric tt*-bundle (E, D, S, g, or) *is a unimodular metric tt*-bundle endowed with an orientation* or *of the bundle* E*.*

Remarks: 1) In special cases, particularly emphasized in the literature, such as the moduli spaces of topological quantum field theories [CV, D] and the moduli spaces of singularities [H], the complexified tt*-bundle $E^{\mathbb{C}}$ is identified with $T^{1,0}M$ and the metric g is positive definite. Here we will consider the case $E = TM$, and hence $E^{\mathbb{C}} = T^{1,0}M + T^{0,1}M$. This includes special complex and special Kähler manifolds, as we shall see.

2) If (E, D, S) is a tt*-bundle then (E, D, S^θ) is a tt*-bundle for all $\theta \in \mathbb{R}$, where

$$S^\theta := D^\theta - D = (\cos\theta) S + (\sin\theta) S_J\,. \tag{1.4}$$

The same remark applies to metric tt*-bundles.
3) Notice that an oriented unimodular metric tt*-bundle (E, D, S, g, or) carries a canonical metric volume element $\nu \in \Gamma(\wedge^r E^*)$, $r = \operatorname{rk} E$, determined by g and or, which is D^θ parallel for all $\theta \in \mathbb{R}$.

Proposition 1 *Let $E \to M$ be a real vector bundle over a complex manifold (M, J) such that E is endowed with a connection D and a section $S \in \Gamma(T^*M \otimes \operatorname{End} E)$. Then (E, D, S) is a tt*-bundle if and only if the following equations are satisfied*

(i) $d^D S = d^D S_J = 0$, *where S and S_J are considered as one-forms with values in* $\operatorname{End} E$ *and d^D is the exterior covariant derivative defined by D,*

(ii) $[S_X, S_Y] = [S_{JX}, S_{JY}]$ *for all X and Y,*

(iii) $R^D(X, Y) + [S_X, S_Y] = 0$ *for all X and Y.*

Proof: Using the relations $2\cos\theta\sin\theta = \sin 2\theta$, $2\cos^2\theta = 1 + \cos 2\theta$ and $2\sin^2\theta = 1 - \cos 2\theta$, we obtain a (finite) Fourier decomposition of R^{D^θ} in the variable θ. The tt*-equation $R^{D^\theta} = 0$ shows that all Fourier components are zero. This yields (i-iii). $\square$

Definition 2 *Let (M, J) be a complex manifold and (N, h) a pseudo-Riemannian manifold. A map $f : M \to N$ is called* pluriharmonic *if $f|_C$ is harmonic for all complex curves $C \subset M$.*

Notice that the harmonicity of $f|_C$ is independent of the choice of a Riemannian metric in the conformal class of C, by conformal invariance of the harmonic map equation for (real) surfaces.

Proposition 2 *Let (M, J) be a complex manifold and (N, h) a pseudo-Riemannian manifold with Levi-Civita connection ∇^h, D a connection on M which satisfies*

$$D_{JY} X = J D_Y X \tag{1.5}$$

for all vector fields which satisfy $\mathcal{L}_X J = 0$ (i.e. for which $X - iJX$ is holomorphic) and ∇ the connection on $T^*M \otimes f^*TN$ which is induced by D and ∇^h.

(i) A map $f : M \to N$ is pluriharmonic if and only if it satisfies the following equation

$$\nabla'' \partial f = 0\,, \tag{1.6}$$

where $\partial f = df^{1,0} \in \Gamma(\bigwedge^{1,0} T^*M \otimes_{\mathbb{C}} (TN)^{\mathbb{C}})$ is the $(1,0)$-component of df and ∇'' is the $(0,1)$-component of $\nabla = \nabla' + \nabla''$.

(ii) Any complex manifold (M, J) admits a torsion-free complex connection, i.e. a torsion-free connection D which satisfies $DJ = 0$.

(iii) Any torsion-free complex connection D satisfies (1.5).

Proof: (i) The condition (1.5) means that $D''Z = 0$ for all local holomorphic vector fields Z, i.e. $\Gamma^{\gamma}_{\bar{\alpha}\beta} = \Gamma^{\bar{\gamma}}_{\bar{\alpha}\beta} = 0$ in terms of the Christoffel symbols of D with respect to holomorphic coordinates z^α. This implies that the Christoffel symbols of D do not contribute to the equation (1.6). The equation is therefore independent of the choice of connection D. In fact, it is straightforward to check that the restriction of (1.6) to every complex curve C reduces to the harmonic map equation for $f|_C : C \to N$.
(ii) is well known, see [KN].
(iii) The conditions $T^D = 0$ and $DJ = 0$ imply that

$$D_{JY}X - JD_YX = [JY, X] + D_X(JY) - JD_YX = [JY, X] + J[X, Y] = -(\mathcal{L}_X J)Y \,. \quad (1.7)$$

The right-hand side vanishes if $\mathcal{L}_X J = 0$. □

Given a Hermitian metric γ on $T^{1,0}M$, or, more generally, a pseudo-Hermitian metric, the *Chern connection* of γ is the unique Hermitian connection $\mathcal{D}$ in the holomorphic bundle $T^{1,0}M$ which satisfies $\mathcal{D}''Z = 0$ for all holomorphic local sections Z of $T^{1,0}M$. The last property is usually written as $\mathcal{D}'' = \bar{\partial}$.

Proposition 3 *Let (M, J) be a complex manifold and $\mathcal{D}$ the Chern connection of a pseudo-Hermitian metric γ on $T^{1,0}M$. Then there is a unique connection D in the real tangent bundle TM such that $DZ = \mathcal{D}Z$ for all local sections Z of $T^{1,0}M$, where D has been complex bilinearly extended to a connection on the complexified tangent bundle. The connection D satisfies (1.5), $DJ = 0$ and $Dg = 0$, where g is the J-invariant pseudo-Riemannian metric defined by*

$$g(X, X) = 2\gamma(X^{1,0}, X^{1,0}) \,, \quad X^{1,0} := \frac{1}{2}(X - iJX) \,, \quad (1.8)$$

for all $X \in TM$.

Conversely, let g be a J-invariant pseudo-Riemannian metric on a complex manifold (M, J). Then there exists a unique connection D in TM, which satisfies the conditions (1.5), $DJ = 0$ and $Dg = 0$. Moreover, D induces a connection in $T^{1,0}M$, which is the Chern connection of the pseudo-Hermitian metric γ on $T^{1,0}M$ defined by (1.8).

The factor 2 is chosen such that γ coincides with the restriction to $T^{1,0}M$ of the sesquilinear extension of g to the complexified tangent bundle.

Proof: We define a connection D in the complexified tangent bundle $(TM)^{\mathbb{C}}$ by

$$D_X Z := \mathcal{D}_X Z \quad \text{and} \quad D_X \bar{Z} := \overline{\mathcal{D}_{\bar{X}} Z} \quad (1.9)$$

for all local sections X of $(TM)^{\mathbb{C}}$ and Z of $T^{1,0}M$. By construction D is real, i.e. is the complex bilinear extension of a connection in TM, which we denote by the same symbol D. Obviously, it is the only real connection such that $DZ = \mathcal{D}Z$ for all local sections Z of $T^{1,0}M$. The equation (1.5) follows from $\mathcal{D}'' = \bar{\partial}$. By construction, D preserves the decomposition $(TM)^{\mathbb{C}} = T^{1,0}M + T^{0,1}M$. Therefore, $DJ = 0$. Finally, $Dg = 0$ follows from the fact that $\mathcal{D}$ is Hermitian.

Conversely, let (M, J, g) be a pseudo-Hermitian manifold. Then we can define a pseudo-Hermitian metric γ in $T^{1,0}M$ by (1.8) and consider its Chern connection $\mathcal{D}$. As we know, it induces a connection D in TM which satisfies (1.5), $DJ = 0$ and $Dg = 0$. To prove the uniqueness, let $\tilde{D}$ be an other connection satisfying (1.5), $\tilde{D}J = 0$ and $\tilde{D}g = 0$. $\tilde{D}$ induces a connection $\tilde{\mathcal{D}}$ in $T^{1,0}M$, which satisfies $\tilde{\mathcal{D}}'' = \bar{\partial}$, due to (1.5), and which is Hermitian with respect to γ. Therefore, $\tilde{\mathcal{D}}$ is the Chern connection of γ, i.e. $\tilde{\mathcal{D}} = \mathcal{D}$. This implies $D = \tilde{D}$, by the first part of the proof. □

Given a metric tt*-bundle (E, D, S, g), we consider the flat connection D^θ for $\theta = 0$: $\nabla := D^0$. Any parallel frame $s = (s_1, \ldots, s_r)$ of E with respect to ∇ defines a map

$$\begin{aligned} &G = G^{(s)} : M \to \mathrm{Sym}_{p,q}(\mathbb{R}^r) = \{A \in \mathrm{GL}(r) | A^t = A \quad \text{has signature } (p,q)\} \\ &x \mapsto G(x) := (g_x(s_i(x), s_j(x))) \,, \end{aligned} \tag{1.10}$$

where (p, q) is the signature of the metric g.

Similarly, for an oriented unimodular metric tt*-bundle (E, D, S, g, ν) with canonical volume element ν and a ∇-parallel frame $s = (s_1, \ldots, s_r)$ such that $\nu(s_1, s_2, \ldots, s_r) = 1$ we have a map

$$G = G^{(s)} : M \to \mathrm{Sym}^1_{p,q}(\mathbb{R}^r) = \{A \in \mathrm{Sym}_{p,q}(\mathbb{R}^r) | \det A = (-1)^q\} \,. \tag{1.11}$$

By Sylvester's Theorem, the general linear group $\mathrm{GL}(r)$ acts transitively on the manifold $\mathrm{Sym}_{p,q}(\mathbb{R}^r)$, which we can identify with the pseudo-Riemannian symmetric space

$$S(p, q) := \mathrm{GL}(r)/\mathrm{O}(p, q) \,. \tag{1.12}$$

The subgroup $\mathrm{O}(p, q) \subset \mathrm{GL}(r)$ is the stabilizer of the matrix $I_{p,q} = \mathrm{diag}(\mathbb{1}_p, -\mathbb{1}_q)$. We shall identify the tangent space of the coset space $S(p, q)$ at the canonical base point $o = e\mathrm{O}(p, q)$ with the vector space

$$\mathrm{sym}(p, q) := \{A \in \mathfrak{gl}(r) | \eta(A\cdot, \cdot) = \eta(\cdot, A\cdot)\} \tag{1.13}$$

of symmetric endomorphisms of $\mathbb{R}^r$ with respect to the standard scalar product $\eta = \eta_{p,q}$ of signature (p, q), which is represented by the matrix $I_{p,q}$. The structure of a symmetric space is defined by the symmetric decomposition

$$\mathfrak{gl}(r) = \mathfrak{o}(p, q) + \mathrm{sym}(p, q) \,. \tag{1.14}$$

The pseudo-Riemannian metric is defined by an $\mathrm{O}(p, q)$-invariant pseudo-Euclidean scalar product on $\mathrm{sym}(p, q)$. For instance, we may choose the metric induced by the trace form:

$$\mathfrak{gl}(r) \ni (X, Y) \mapsto \mathrm{tr}\, XY \,. \tag{1.15}$$

Similarly, $\mathrm{SL}(r)$ acts transitively on the manifold $\mathrm{Sym}^1_{p,q}(\mathbb{R}^r)$, which we can identify with the pseudo-Riemannian symmetric space

$$S_1(p, q) := \mathrm{SL}(r)/\mathrm{SO}(p, q) \,. \tag{1.16}$$

We have the de Rham decomposition

$$S(p, q) = \mathbb{R} \times S_1(p, q) \,, \tag{1.17}$$

where the flat factor corresponds to the connected central subgroup

$$\mathbb{R}^{>0} = \{\lambda \mathrm{Id} | \lambda > 0\} \subset \mathrm{GL}(r) \tag{1.18}$$

and the other factor is always indecomposable and even irreducible if $(p, q) \neq (1, 1)$. The tangent space of $\mathrm{SL}(r)/\mathrm{SO}(p, q)$ at the canonical base point $o = e\mathrm{SO}(p, q)$ is identified with the trace-free η-symmetric matrices:

$$\mathrm{sym}_0(p, q) := \{A \in \mathrm{sym}(p, q) | \mathrm{tr}\, A = 0\}\,. \tag{1.19}$$

Under a change of parallel (respectively, parallel unimodular) frame $s \to su$, $u \in \mathrm{GL}(r)$ (respectively, $u \in \mathrm{SL}(r)$), the map $G = G^{(s)}$ transforms as

$$G^{(su)} = u^{-1} \cdot G^{(s)} = u^t G^{(s)} u\,, \tag{1.20}$$

where the dot stands for the action of $\mathrm{GL}(r)$ on $\mathrm{Sym}_{p,q}(\mathbb{R}^r)$.

The following theorem is proven in [S2], cf. [S1]. In the case where $E^{\mathbb{C}} = T^{1,0}M$ and the metric g is positive definite it is due to Dubrovin [D].

Theorem 1 *Let (E, D, S, g) be a metric tt*-bundle over a simply connected complex manifold M. Then the map*

$$G^{(s)} = (g(s_i, s_j)) : M \to \mathrm{Sym}_{p,q}(\mathbb{R}^r) \cong \mathrm{GL}(r)/\mathrm{O}(p, q) = S(p, q) \tag{1.21}$$

associated to a parallel frame $s = (s_1, \ldots, s_r)$ of E with respect to the flat connection $\nabla = D^0$ is pluriharmonic. Moreover, for all $x \in M$, the image of $T^{1,0}_x M \subset (T_x M) \otimes \mathbb{C}$ under the complex linear extension of $dL_u^{-1} dG_x : T_x M \to T_o S(p, q) = \mathrm{sym}(p, q)$ consists of commuting matrices, where $u \in \mathrm{GL}(r)$ is any element such that $G(x) = u \cdot o$ and $L_u : S(p, q) \to S(p, q)$ is the isometry of $S(p, q)$ induced by the left-multiplication by u in $\mathrm{GL}(r)$.

Conversely, let M be a simply connected complex manifold and $f : M \to \mathrm{Sym}_{p,q}(\mathbb{R}^r) \cong S(p, q)$ a pluriharmonic map such that, for all $x \in M$, the image of $T^{1,0}_x M$ under the complex linear extension of $dL_u^{-1} df_x : T_x M \to T_o S(p, q) = \mathrm{sym}(p, q)$ consists of commuting matrices, where $u \in \mathrm{GL}(r)$ is any element such that $f(x) = u \cdot o$. Then there exists a metric tt-bundle (E, D, S, g) over M and a parallel frame s such that $f = G^{(s)}$. The condition on the image of $T^{1,0}_x M$ is automatically satisfied if $pq = 0$, which corresponds to a positive or negative definite metric g.*

The same correspondence holds for oriented unimodular tt-bundles and pluriharmonic maps into $\mathrm{Sym}^1_{p,q}(\mathbb{R}^r) \cong \mathrm{SL}(r)/\mathrm{SO}(p, q) = S_1(p, q)$.*

Now we shall explain in more detail the condition on the image of $T^{1,0}M$ under the differential of f in the theorem. Above we have always identified $\mathrm{Sym}_{p,q}(\mathbb{R}^r)$ with $S(p, q)$. Let us denote by

$$\varphi : \mathrm{Sym}_{p,q}(\mathbb{R}^r) \to S(p, q)\,, \quad S \mapsto \tilde{S} = \varphi(S)\,, \tag{1.22}$$

that identification, which is $\mathrm{GL}(r)$-equivariant and maps $I = I_{p,q}$ to the canonical base point o. We can identify the tangent space $T_S \mathrm{Sym}_{p,q}(\mathbb{R}^r)$ at $S \in \mathrm{Sym}_{p,q}(\mathbb{R}^r)$ with the (ambient) vector space of symmetric matrices:

$$T_S \mathrm{Sym}_{p,q}(\mathbb{R}^r) = \mathrm{Sym}(\mathbb{R}^r) := \{A \in \mathrm{Mat}(r, \mathbb{R}) | A^t = A\}\,. \tag{1.23}$$

As above for $S = I$, the tangent space $T_{\tilde{S}}S(p,q)$ is canonically identified with the vector space of S-symmetric matrices:

$$T_{\tilde{S}}S(p,q) = \mathrm{sym}(S) := \{A \in \mathfrak{gl}(r) | A^t S = SA\} . \tag{1.24}$$

Note that $\mathrm{sym}(I_{p,q}) = \mathrm{sym}(p,q)$.

Proposition 4 *The differential of φ at $S \in \mathrm{Sym}_{p,q}(\mathbb{R}^r)$ is given by*

$$\mathrm{Sym}(\mathbb{R}^r) \ni X \mapsto -\frac{1}{2}S^{-1}X \in S^{-1}\mathrm{Sym}(\mathbb{R}^r) = \mathrm{sym}(S) . \tag{1.25}$$

Let us now consider the differential

$$df_x : T_xM \to \mathrm{Sym}(\mathbb{R}^r) \tag{1.26}$$

of $f : M \to \mathrm{Sym}_{p,q}(\mathbb{R}^r)$ at $x \in M$ and the differential

$$d\tilde{f}_x : T_xM \to \mathrm{sym}(f(x)) \tag{1.27}$$

of $\tilde{f} = \varphi \circ f : M \to S(p,q)$. Then the condition on the image of the differential of f in the theorem is that

$$dL_u^{-1} d\tilde{f}(T_x^{1,0}M) \subset \mathrm{sym}(p,q) \otimes \mathbb{C} \quad \text{consists of commuting matrices} , \tag{1.28}$$

where $\tilde{f}(x) = uo$. This is equivalent to the condition that $d\tilde{f}(T_x^{1,0}M) \subset \mathrm{sym}(\tilde{f}(x)) \otimes \mathbb{C}$ consists of commuting matrices. This follows from the fact that

$$dL_u : T_oS(p,q) \to T_{uo}S(p,q) = T_{\tilde{f}(x)}S(p,q) \tag{1.29}$$

corresponds to

$$Ad_u : \mathrm{sym}(p,q) = \mathrm{sym}(I) \to \mathrm{sym}(u \cdot I) = \mathrm{sym}(\tilde{f}(x)) \tag{1.30}$$

and that the adjoint representation preserves the Lie bracket.

Finally, $d\tilde{f}_x = d\varphi \, df_x = -\frac{1}{2} f(x)^{-1} df_x$ and, therefore,

$$d\tilde{f}(T_x^{1,0}M) = f(x)^{-1} df_x(T_x^{1,0}M) . \tag{1.31}$$

This shows that f satisfies the condition (1.28) if and only if the matrices $f(x)^{-1}df_x(Z)$ and $f(x)^{-1}df_x(W)$ commute for all Z, $W \in T_x^{1,0}M$. This is equivalent to

$$[f(x)^{-1}df_x(JX), f(x)^{-1}df_x(JY)] = [f(x)^{-1}df_x(X), f(x)^{-1}df_x(Y)] \tag{1.32}$$

for all X, $Y \in T_xM$.

2 Special complex and special Kähler manifolds

In this section we recall some basic results on special complex manifolds and special Kähler manifolds. For more detailed information the reader is referred to [ACD], see also [F].

Definition 3 *A* special complex manifold (M, J, ∇) *is a complex manifold* (M, J) *endowed with a flat torsion-free connection* ∇ *(on the real tangent-bundle) such that* ∇J *is symmetric.*
A special Kähler manifold (M, J, ∇, ω) *is a special complex manifold* (M, J, ∇) *endowed with a* J*-invariant and* ∇*-parallel symplectic form* ω*. The (pseudo)-Kähler-metric* $g(\cdot,\cdot) = \omega(J\cdot,\cdot)$ *is called the* special Kähler metric *of the special Kähler manifold* (M, J, ∇, ω)*.*

Given a complex manifold (M, J) with a flat connection ∇, we define its conjugate connection by

$$\nabla^J_X = \nabla_X - J\nabla_X J \text{ with } X \in TM. \tag{2.1}$$

On a special complex manifold (M, J, ∇) the connection ∇^J is torsion-free. In addition, one can introduce a torsion-free connection

$$D := \frac{1}{2}(\nabla + \nabla^J) = \nabla - S, \text{ where } S := \frac{1}{2}J\nabla J, \tag{2.2}$$

which satifies $DJ = 0$, as follows from a short calculation.
In the case of a special Kähler manifold (M, J, ∇, ω) the connection D is the Levi-Civita connection of the special Kähler metric g and the endomorphism-field S anticommutes with the complex structure J, i.e. :

$$JS_X = -S_X J \quad \text{for all} \quad X \in TM\,. \tag{2.3}$$

Now we explain part of the extrinsic construction of special Kähler-manifolds given in [ACD]. In order to do this, we consider the complex vector space $V = T^*\mathbb{C}^n = \mathbb{C}^{2n}$ with canonical coordinates $(z^1, \ldots, z^n, w_1, \ldots, w_n)$ endowed with the standard complex symplectic form $\Omega = \sum_{i=1}^n dz^i \wedge dw_i$ and the standard real structure $\tau : V \to V$ with fixed points $V^\tau = T^*\mathbb{R}^n$. These define a Hermitian form $\gamma := i\Omega(\cdot, \tau\cdot)$.
Let (M, J) be a complex manifold (M, J) of complex dimension n. We call a holomorphic immersion $\phi : M \to V$ nondegenerate (respectively Lagrangian) if $\phi^*\gamma$ is nondegenerate (respectively, if $\phi^*\Omega = 0$). If ϕ is nondegenerate it defines a, possibly indefinite, Kähler metric $g = Re\,\phi^*\gamma$ on the complex manifold (M, J) and the corresponding Kähler form $g(\cdot, J\cdot)$ is a $J-$invariant symplectic form.

The following theorem gives a description of simply connected special Kähler-manifolds in terms of the above data:

Theorem 2 *[ACD] Let* (M, J, ∇, ω) *be a simply connected special Kähler manifold of complex dimension* n*, then there exists a holomorphic nondegenerate Lagrangian immersion* $\phi : M \to V = T^*\mathbb{C}^n$ *inducing the Kähler metric* g*, the connection* ∇ *and the symplectic form* $\omega = 2\phi^*\left(\sum_{i=1}^n dx^i \wedge dy_i\right) = g(\cdot, J\cdot)$ *on* M*. Moreover,* ϕ *is unique up to an affine transformation of* V *preserving the complex symplectic form* Ω *and the real structure* τ*. The flat connection* ∇ *is completely determined by the condition* $\nabla\phi^*dx^i = \nabla\phi^*dy_i = 0$*,* $i = 1, \ldots, n,$ *where* $x^i = Re\, z^i$ *and* $y_i = Re\, w_i$*.*

3 Special complex and special Kähler manifolds as solutions of the tt*-equations

Let (E, D, S) be a tt*-bundle over a complex manifold (M, J). We are now interested in the case $E = TM$. In that case it is natural to consider tt*-bundles for which the connection $D^\theta = D + (\cos\theta)S + (\sin\theta)S_J$ is torsion-free.

Definition 4 *A tt*-bundle (TM, D, S) over a complex manifold (M, J) is called* special *if D^θ is torsion-free and special, i.e. $D^\theta J$ is symmetric for all θ.*

Proposition 5 *A tt*-bundle (TM, D, S) is special if and only if D is torsion-free and DJ, S and S_J are symmetric.*

Proof: The torsion T^θ of D^θ is given by

$$T^\theta(X, Y) = T(X, Y) + \cos\theta(S_X Y - S_Y X) + \sin\theta(S_{JX} Y - S_{JY} X)\,, \tag{3.1}$$

where T is the torsion of D. This shows that $T^\theta = 0$ for all θ if and only if $T = 0$ and S and S_J are symmetric. The equation

$$(D^\theta_X J)Y = (D_X J)Y + \cos\theta[S_X, J]Y + \sin\theta[S_{JX}, J]Y \tag{3.2}$$

shows that $D^\theta J$ is symmetric if DJ, S and S_J are symmetric. Conversely, if $T^\theta = 0$ and $D^\theta J$ is symmetric, then, by the first part of the proof, S and S_J are symmetric and equation (3.2) shows that DJ is symmetric. □

Theorem 3

(i) Let (M, J, ∇) be a special complex manifold. Put $S := \frac{1}{2}J\nabla J$ and $D := \nabla - S$. Then (TM, D, S) is a special tt-bundle, which satisfies the additional conditions:*
a) $S_X J = -JS_X$ for all $X \in TM$ and
b) $DJ = 0$.
This defines a map Φ from special complex manifolds to special tt-bundles.*

(ii) Let (TM, D, S) be a special tt-bundle over a complex manifold (M, J). Then $(M, J, \nabla := D + S)$ is a special complex manifold. This defines a map Ψ from special tt*-bundles to special complex manifolds such that $\Psi \circ \Phi = \mathrm{Id}$. If (TM, D, S) is a special tt*-bundle satisfying the conditions a) and b) in (i), then $\Phi(\Psi(TM, D, S)) = (TM, D, S)$.*

(iii) Let (M, J, g, ∇) be a special Kähler manifold with S and D defined as in (i). Then (TM, D, S, g) is a special metric tt-bundle. This defines a map Φ from special Kähler manifolds to special metric tt*-bundles.*

(iv) Let (TM, D, S, g) be a special metric tt-bundle over a pseudo-Hermitian manifold (M, J, g) satisfying the conditions a) and b) in (i). Then $(M, J, \nabla := D + S, g)$ is a special Kähler manifold. In particular, we have a map Ψ from special metric tt*-bundles over pseudo-Hermitian manifolds satisfying a) and b) to special Kähler manifolds. Moreover, Ψ is a bijection and $\Psi^{-1} = \Phi$.*

(v) *Let (TM, D, S, g) be a metric tt*-bundle over a pseudo-Hermitian manifold (M, J, g) satisfying a) and b) in (i) and such that D is torsion-free. Then it is special if and only if $(M, J, \nabla := D + S, g)$ is a special Kähler manifold.*

Proof: (i) Let (M, J, ∇) be a special complex manifold with S and D defined as above. Then

$$\nabla^\theta = e^{\theta J} \circ \nabla \circ e^{-\theta J} \tag{3.3}$$

is a family of flat torsion-free special connections. Using $\nabla = D + S$ and (2.3) we can write

$$\nabla^\theta_X = D_X + e^{2\theta J} S_X \,. \tag{3.4}$$

The following calculation shows that $\nabla^\theta = D^{-2\theta}$, where D^θ is defined in (1.2):

$$\begin{aligned} \nabla^\theta_X - D_X &= e^{2\theta J} S_X = \cos(2\theta) S_X + \sin(2\theta) J S_X \\ &\overset{(*)}{=} \cos(-2\theta) S_X + \sin(-2\theta) S_{JX} = D^{-2\theta}_X - D_X \,, \quad X \in TM \,. \end{aligned}$$

At $(*)$ we have used that $JS_X = -S_{JX}$, which follows from

$$JS_X Y = JS_Y X = -S_Y JX = -S_{JX} Y \,, \quad X, Y \in T_pM \,. \tag{3.5}$$

Here we used the symmetry of S and (2.3). This shows that (TM, D, S) is a special tt*-bundle.

(ii) Let (TM, D, S) be a special tt*-bundle. This means that D^θ is flat, torsion-free and special. In particular, $\nabla = D + S = D^0$ is flat, torsion-free and special and (M, J, ∇) is a special complex manifold. It is clear that $\Psi \circ \Phi = \mathrm{Id}$.

Conversely, let (TM, D, S) be a special tt*-bundle such that $DJ = 0$ and $S_X J = -JS_X$ for all X. Then we can recover D and S from $\nabla = D + S$ by the formulas $S = \frac{1}{2} J \nabla J$ and $D = \nabla - S$. In fact, Let (TM, D', S') be an other special tt*-bundle over (M, J) such that $D'J = 0$ and $S'_X J = -JS'_X$ for all $X \in TM$ and $\nabla = D + S = D' + S'$. Then

$$0 = D'_X J = \nabla_X J - [S'_X, J] = \nabla_X J + 2JS'_X \tag{3.6}$$

for all $X \in TM$. This shows that $S'_X = \frac{1}{2} J \nabla_X J = S_X$ and $D' = \nabla - S' = \nabla - S = D$.

(iii) Let (M, J, g, ∇) be a special Kähler manifold with S and D defined as in (i). Then, by (i), (TM, D, S) is a special tt*-bundle and satifies a) and b). To prove that it is a *metric* tt*-bundle we have to check that $Dg = 0$ and that (1.3) is satisfied. Since $DJ = 0$, by b), the equation $Dg = 0$ is equivalent to the following claim:

Claim: The Kähler form ω is D-parallel; $D\omega = 0$.

In fact $\nabla\omega = 0$ and $S_X = \frac{1}{2} J \nabla_X J$, $X \in TM$, is the product of two anticommuting ω-skew-symmetric endomorphisms $A = \frac{1}{2} J$ and $B = \nabla_X J$. This implies that S_X is ω-skew-symmetric and, thus, $D\omega = 0$.

The endomorphism S_X is ω-skew-symmetric and anticommutes with J, by a). Therefore S_X is symmetric with respect to $g = \omega(J\cdot, \cdot)$.

(iv) Let (TM, D, S, g) be a special metric tt*-bundle over a pseudo-Hermitian manifold (M, J, g) satisfying a) and b) in (i). Thanks to (ii), we know already that $(M, J, \nabla := D + S)$ is a special complex manifold. Therefore it suffices to prove that $\nabla\omega = 0$. The assumption $Dg = 0$ and property b) imply that $D\omega = 0$. Now it is sufficient to observe that the endomorphisms S_X, $X \in TM$, are ω-skew-symmetric. In fact, S_X is g-symmetric in virtue of (1.3) and anticommutes with J, by a). This shows that (M, J, ∇, g) is a special Kähler manifold. The remaining statements follow from (ii).

(v) Let (TM, D, S, g) be a metric tt*-bundle over a pseudo-Hermitian manifold (M, J, g) such that $(M, J, \nabla := D + S, g)$ is a special Kähler manifold. If D is torsion-free, then it is the Levi-Civita connection of g and, thus, $D = \nabla - \frac{1}{2}J\nabla J$, see section 2. Now we can conclude that $\Phi(M, J, \nabla, g) = (TM, D, S, g)$. This shows that (TM, D, S, g) is a special metric tt*-bundle. The converse follows from (iv). □

Corollary 1 *Any special metric tt*-bundle (TM, D, S, g) over a pseudo-Hermitian manifold (M, J, g) which satisfies a) and b) in Theorem 3 (i) is oriented and unimodular.*

Proof: TM is canonically oriented by the complex structure J. By Theorem 3, $(M, J, g, \nabla = D + S)$ is a special Kähler manifold. Its Kähler form is parallel with respect to D and ∇ and hence invariant under $S_X = \nabla_X - D_X$ for all $X \in TM$. This shows that $\mathrm{tr}\, S_X = 0$. □

In [H] special complex and special Kähler geometry is interpreted in terms of variations of Hodge structure of weight 1 on the complexified tangent bundle. From this interpretation and his discussion of tt*-geometry, it follows that any special complex (respectively, special Kähler) manifold defines a tt*-bundle (respectively, a metric tt*-bundle) in the sense of our definition.

4 The pluriharmonic map in the case of a special Kähler manifold

4.1 The Gauß maps of a special Kähler manifold

Let (M, J, g, ∇) be a special Kähler manifold of complex dimension $n = k + l$ and of Hermitian signature (k, l), i.e. g has signature $(2k, 2l)$. Let $(\widetilde{M}, J, g, \nabla)$ be its universal covering with the pullback special Kähler structure, which is again denoted by (J, g, ∇). According to Theorem 2, there exists a (holomorphic) Kählerian Lagrangian immersion $\phi : \widetilde{M} \to V = T^*\mathbb{C}^n = \mathbb{C}^{2n}$, which is unique up to a complex affine transformation of V with linear part in $\mathrm{Sp}(\mathbb{R}^{2n})$. We consider the *dual Gauß map* of ϕ

$$L : \widetilde{M} \to Gr_0^{k,l}(\mathbb{C}^{2n}), \quad p \mapsto L(p) := T_{\phi(p)}\widetilde{M} := d\phi_p T_p\widetilde{M} \subset V \tag{4.1}$$

into the Grassmannian of complex Lagrangian subspaces $W \subset V$ of *signature (k,l)*, i.e. such that the restriction of γ to W is a Hermitian form of signature (k, l). The map $L : \widetilde{M} \to Gr_0^{k,l}(\mathbb{C}^{2n})$ is in fact the dual of the *Gauß map*

$$L^\perp : \widetilde{M} \to Gr_0^{l,k}(\mathbb{C}^{2n}), \quad p \mapsto L(p)^\perp = \overline{L(p)} \cong L(p)^* \,. \tag{4.2}$$

Here $L(p)^\perp$ stands for the γ-orthogonal complement of $L(p)$ and the isomorphism $\overline{L(p)} \cong L(p)^*$ is induced by the symplectic form Ω on $V = L(p) \oplus \overline{L(p)}$.

The Grassmannian $Gr_0^{k,l}(\mathbb{C}^{2n})$ is an open subset of the complex Grassmannian $Gr_0(\mathbb{C}^{2n})$ of complex Lagrangian subspaces $W \subset V$ and hence a complex submanifold.

Proposition 6 *(i) The dual Gauß map $L : \widetilde{M} \to Gr_0^{k,l}(\mathbb{C}^{2n})$ is holomorphic*

(ii) The Gauß map $L^\perp : \widetilde{M} \to Gr_0^{l,k}(\mathbb{C}^{2n})$ is antiholomorphic.

Proof: The holomorphicity of L follows from that of ϕ. Part (ii) follows from (i), since $L^\perp = \overline{L} : p \mapsto \overline{L(p)}$. □

The real symplectic group $\mathrm{Sp}(\mathbb{R}^{2n})$ acts transitively on $Gr_0^{k,l}(\mathbb{C}^{2n})$ and we have the following identification:

$$Gr_0^{k,l}(\mathbb{C}^{2n}) = \mathrm{Sp}(\mathbb{R}^{2n})/\mathrm{U}(k,l)\,. \tag{4.3}$$

Here $\mathrm{U}(k,l) \subset \mathrm{Sp}(\mathbb{R}^{2n})$ is defined as the stabilizer of

$$W_o = \mathrm{span}\{\frac{\partial}{\partial z^1} + i\frac{\partial}{\partial w_1}, \cdots, \frac{\partial}{\partial z^k} + i\frac{\partial}{\partial w_k}, \frac{\partial}{\partial z^{k+1}} - i\frac{\partial}{\partial w_{k+1}}, \cdots, \frac{\partial}{\partial z^n} - i\frac{\partial}{\partial w_n}\}\,. \tag{4.4}$$

The Gauß maps L and $L^\perp$ induce Gauß maps

$$L_M \;:\; M \to \Gamma \setminus Gr_0^{k,l}(\mathbb{C}^{2n}) \tag{4.5}$$

$$L_M^\perp \;:\; M \to \Gamma \setminus Gr_0^{l,k}(\mathbb{C}^{2n}) \tag{4.6}$$

into the quotient of the Grassmannian by the holonomy group $\Gamma = \mathrm{Hol}(\nabla) \subset \mathrm{Sp}(\mathbb{R}^{2n})$ of the flat symplectic connection ∇.

Corollary 2 *(i) The dual Gauß map $L_M : M \to \Gamma \setminus Gr_0^{k,l}(\mathbb{C}^{2n})$ of M is holomorphic.*

(ii) The Gauß map $L_M^\perp : M \to \Gamma \setminus Gr_0^{l,k}(\mathbb{C}^{2n})$ is antiholomorphic.

The Grassmannian $Gr_0^{k,l}(\mathbb{C}^{2n})$ is a pseudo-Hermitian symmetric space and, in particular, a homogeneous pseudo-Kähler manifold. If $\Gamma \subset \mathrm{Sp}(\mathbb{R}^{2n})$ acts properly discontinuously on $Gr_0^{k,l}(\mathbb{C}^{2n})$ then $\Gamma \setminus Gr_0^{k,l}(\mathbb{C}^{2n})$ is a locally symmetric space of pseudo-Hermitian type.

4.2 Holomorphic coordinates on the Grassmannian $Gr_0^{k,l}(\mathbb{C}^{2n})$ of complex Lagrangian subspaces of signature (k,l)

In this section we shall introduce a local model for the Grassmannian $Gr_0^{k,l}(\mathbb{C}^{2n})$ and determine the corresponding local expression for the dual Gauß map. This model is a pseudo-Riemannian analogue of the Siegel upper half-space

$$\mathrm{Sym}^+(\mathbb{C}^n) := \{A \in \mathrm{Mat}(n,\mathbb{C}) | A^t = A \quad \text{and} \quad \mathrm{Im}\, A \quad \text{is positive definite}\}\,. \tag{4.7}$$

Our aim is to construct holomorphic coordinates for the complex manifold $Gr_0^{k,l}(\mathbb{C}^{2n})$ in a Zariski-open neighborhood of a point W_0 of the Grassmannian represented by a

Lagrangian subspace $W_0 \subset V$ of signature (k,l). Using a transformation from $\mathrm{Sp}(\mathbb{R}^{2n})$ we can assume that $W_0 = W_o$, see (4.4). Let $U_0 \subset Gr_0^{k,l}(\mathbb{C}^{2n})$ be the open subset consisting of $W \in Gr_0^{k,l}(\mathbb{C}^{2n})$ such that the projection

$$\pi_{(z)} : V = T^*\mathbb{C}^n = \mathbb{C}^n \oplus (\mathbb{C}^n)^* \to \mathbb{C}^n \tag{4.8}$$

onto the first summand (z-space) induces an isomorphism

$$\pi_{(z)}|_W : W \xrightarrow{\sim} \mathbb{C}^n \,. \tag{4.9}$$

Notice that $U_0 \subset Gr_0^{k,l}(\mathbb{C}^{2n})$ is an open neighborhood of the base point W_0. For elements $W \in U_0$ we can express w_i as a function of $z = (z^1, \ldots, z^n)$. In fact,

$$w_i = \sum C_{ij} z^j \,, \tag{4.10}$$

where

$$(C_{ij}) \in \mathrm{Sym}_{k,l}(\mathbb{C}^n) = \{A \in \mathrm{Mat}(n,\mathbb{C}) | A^t = A \quad \text{and} \quad \mathrm{Im}\, A \quad \text{has signature } (k,l)\} \,. \tag{4.11}$$

Proposition 7 *The map*

$$C : U_0 \to \mathrm{Sym}_{k,l}(\mathbb{C}^n) \,, \quad W \mapsto C(W) := (C_{ij}) \tag{4.12}$$

is a local holomorphic chart for the Grassmannian $Gr_0^{k,l}(\mathbb{C}^{2n})$.

Remark: The open subset $\mathrm{Sym}_{k,l}(\mathbb{C}^n) \subset \mathrm{Sym}(\mathbb{C}^n) = \{A \in \mathrm{Mat}(n,\mathbb{C}) | A^t = A\}$ is a generalization of the famous Siegel upper half-space $\mathrm{Sym}_{n,0}(\mathbb{C}^n) = \mathrm{Sym}^+(\mathbb{C}^n)$, which is a Siegel domain of type I. In the latter case, we have $U_0 = \mathrm{Sp}(\mathbb{R}^{2n})/\mathrm{U}(n)$ and a global coordinate chart

$$C : Gr_0^{n,0}(\mathbb{C}^{2n}) = \mathrm{Sp}(\mathbb{R}^{2n})/\mathrm{U}(n) \xrightarrow{\sim} \mathrm{Sym}_{n,0}(\mathbb{C}^n) \,. \tag{4.13}$$

We shall now describe the dual Gauß map L in local holomorphic coordinates in neighborhoods of $p_0 \in \widetilde{M}$ and $L(p_0) \in Gr_0^{k,l}(\mathbb{C}^{2n})$. Applying a transformation from $\mathrm{Sp}(\mathbb{R}^{2n})$, if necessary, we can assume that $L(p_0) \in U_0$. We put $U := L^{-1}(U_0)$. The open subset $U \subset \widetilde{M}$ is a neighborhood of p_0.

Let $\phi : \widetilde{M} \to T^*\mathbb{C}^n$ be the Kählerian Lagrangian immersion. It defines a system of local (special) holomorphic coordinates

$$\varphi := \pi_{(z)} \circ \phi|_U : U \xrightarrow{\sim} U' \subset \mathbb{C}^n \,, \quad p \mapsto (z^1(\phi(p)), \cdots, z^n(\phi(p))) \tag{4.14}$$

and we have the following commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{L} & U_0 \\ \varphi \downarrow & & \downarrow C \\ U' & \xrightarrow{L_U} & \mathrm{Sym}_{k,l}(\mathbb{C}^n) \,, \end{array} \tag{4.15}$$

where the vertical arrows are holomorphic diffeomorphisms and L_U at $z = (z^1, \dots z^n)$ is given by

$$L_U(z) = (F_{ij}(z)) := \left(\frac{\partial^2 F(z)}{\partial z^i \partial z^j} \right) . \tag{4.16}$$

Here $F = F(z)$ is a holomorphic function on $U' \subset \mathbb{C}^n$ determined, up to a constant, by the equations

$$w_j(\phi(p)) = \left. \frac{\partial F}{\partial z^j} \right|_{z(\phi(p))} . \tag{4.17}$$

Summarizing, we obtain the following proposition.

Proposition 8 *The dual Gauß map L has the following coordinate expression*

$$L_U = C \circ L \circ \varphi^{-1} = (F_{ij}) , \tag{4.18}$$

where $\varphi : U \to \mathbb{C}^n$ is the (special) holomorphic chart of $\widetilde{M}$ associated to the Kählerian Lagrangian immersion ϕ, see (4.14), and $C : U_0 \to \mathrm{Sym}(\mathbb{C}^n)$ is the holomorphic chart of $Gr_0^{k,l}(\mathbb{C}^{2n})$ constructed in (4.12).

4.3 The special Kähler metric in affine coordinates

As before, let (M, J, g, ∇) be a special Kähler manifold of Hermitian signature (k, l), $k+l = n = \dim_{\mathbb{C}} M$, and $(\widetilde{M}, J, g, \nabla)$ its universal covering. As in section 1, we shall now consider the metric g in a ∇-parallel frame. Such a frame is provided by the Kählerian Lagrangian immersion $\phi : \widetilde{M} \to V$. In fact, any point $p \in \widetilde{M}$ has a neighborhood in which the functions $\widetilde{x}^i := \mathrm{Re}\, z^i \circ \phi$, $\widetilde{y}_i := \mathrm{Re}\, w_i \circ \phi$, $i = 1, \dots, n$, form a system of local ∇-affine coordinates. We recall that the ∇-parallel Kähler form is given by $\omega = 2 \sum d\widetilde{x}^i \wedge d\widetilde{y}_i$. This implies that the globally defined one-forms $\sqrt{2} d\widetilde{x}^i$, $\sqrt{2} d\widetilde{y}_i$ constitute a ∇-parallel unimodular frame

$$(e^a)_{a=1,\dots,2n} = (e^1, \dots, e^{2n}) := (\sqrt{2} d\widetilde{x}^1, \dots, \sqrt{2} d\widetilde{x}^n, \sqrt{2} d\widetilde{y}_1, \dots, \sqrt{2} d\widetilde{y}_n) \tag{4.19}$$

of $T^*\widetilde{M}$ with respect to the metric volume form $\nu = (-1)^{n+1} \frac{\omega^n}{n!} = 2^n d\widetilde{x}^1 \wedge \dots \wedge d\widetilde{y}_n$. The dual frame (e_a) of $T\widetilde{M}$ is also ∇-parallel and unimodular. The metric defines a smooth map

$$G : \widetilde{M} \to \mathrm{Sym}^1_{2k,2l}(\mathbb{R}^{2n}) = \{ A \in \mathrm{Mat}(2n, \mathbb{R}) | A^t = A\, , \ \det A = 1 \text{ has signature } (2k, 2l) \} \tag{4.20}$$

by

$$p \mapsto G(p) := (g_{ab}(p)) := (g_p(e_a, e_b)) . \tag{4.21}$$

We will call $G = (g_{ab})$ *the fundamental matrix* of ϕ. As before, we identify

$$\mathrm{Sym}^1_{2k,2l}(\mathbb{R}^{2n}) = \mathrm{SL}(2n, \mathbb{R}) / \mathrm{SO}(2k, 2l) . \tag{4.22}$$

This is a pseudo-Riemannian symmetric space. For conventional reasons, in this section, $\mathrm{SO}(2k, 2l) \subset \mathrm{SL}(2n, \mathbb{R})$ is defined as the stabilizer of the symmetric matrix

$$E_o := \mathrm{diag}(\mathbb{1}_k, -\mathbb{1}_l, \mathbb{1}_k, -\mathbb{1}_l) . \tag{4.23}$$

The fundamental matrix induces a map

$$G_M : M \to \Gamma \setminus \mathrm{Sym}^1_{2k,2l}(\mathbb{R}^{2n}) \tag{4.24}$$

into the quotient of $\mathrm{Sym}^1_{2k,2l}(\mathbb{R}^{2n})$ by the action of the holonomy group $\Gamma = \mathrm{Hol}(\nabla) \subset \mathrm{Sp}(\mathbb{R}^{2n}) \subset \mathrm{SL}(2n, \mathbb{R})$. The target $\Gamma \setminus \mathrm{Sym}^1_{2k,2l}(\mathbb{R}^{2n})$ is a pseudo-Riemannian locally symmetric space, provided that Γ acts properly discontinuously.

Theorem 4 *The fundamental matrix* $G : \widetilde{M} \to \mathrm{Sym}^1_{2k,2l}(\mathbb{R}^{2n}) = \mathrm{SL}(2n,\mathbb{R})/\mathrm{SO}(2k, 2l)$ *takes values in the totally geodesic submanifold*

$$\iota : Gr_0^{k,l}(\mathbb{C}^{2n}) = \mathrm{Sp}(\mathbb{R}^{2n})/\mathrm{U}(k, l) \hookrightarrow \mathrm{SL}(2n, \mathbb{R})/\mathrm{SO}(2k, 2l) \tag{4.25}$$

and coincides with the dual Gauß map $L : \widetilde{M} \to Gr_0^{k,l}(\mathbb{C}^{2n})$*:* $G = \iota \circ L$.

Proof: The proof will follow from a geometric description of the inclusion ι. To any Lagrangian subspace $W \in Gr_0^{k,l}(\mathbb{C}^{2n})$ we can associate the scalar product $g^W := \mathrm{Re}\,\gamma|_W$ of signature $(2k, 2l)$ on $W \subset V$. The projection onto the real points

$$\mathrm{Re} : V = T^*\mathbb{C}^n \to T^*\mathbb{R}^n = \mathbb{R}^{2n}\,, \quad v \mapsto \mathrm{Re}\, v = \frac{1}{2}(v + \bar{v}) \tag{4.26}$$

induces an isomorphism of real vector spaces $W \xrightarrow{\sim} \mathbb{R}^{2n}$ the inverse of which we denote by $\psi = \psi_W$. We claim that

$$\iota(W) = \psi^* g^W =: (g^W_{ab}) =: G^W\,. \tag{4.27}$$

To check this, it is sufficient to prove that the map

$$Gr_0^{k,l}(\mathbb{C}^{2n}) \ni W \mapsto G^W \in \mathrm{Sym}^1_{2k,2l}(\mathbb{R}^{2n}) \tag{4.28}$$

is $\mathrm{Sp}(\mathbb{R}^{2n})$-equivariant and maps the base point W_o with stabilizer $\mathrm{U}(k, l)$, see (4.4), to the base point E_o with stabilizer $\mathrm{SO}(2k, 2l)$, see (4.23). Let us verify that indeed $G^{W_o} = E_o$.

Using the definition of γ, one finds for the basis

$$(e_j^{\pm}) := \left(\frac{\partial}{\partial z^j} \pm i \frac{\partial}{\partial w_j}\right) \tag{4.29}$$

of V that the only non-vanishing components of γ are $\gamma(e_j^{\pm}, e_j^{\pm}) = \pm 2$. This shows that $g^{W_o} = \mathrm{Re}\,\gamma|_{W_o}$ is represented by the matrix $2E_o$ with respect to the basis

$$(e_1^+, \ldots, e_k^+, e_1^-, \ldots, e_l^-, ie_1^+, \ldots, ie_k^+, ie_1^-, \ldots, ie_l^-)\,. \tag{4.30}$$

In order to calculate $G^{W_o} = (g^{W_o}_{ab}) = (g(\psi e_a, \psi e_b))$, we need to pass from the real basis (4.30) of W_o to the real basis (ψe_a).

Recall that the real structure τ is complex conjugation with respect to the coordinates (z^i, w_i). This implies that

$$\psi^{-1}(e_j^+) = \frac{\partial}{\partial x^j} = \sqrt{2} e_j\,, \quad \psi^{-1}(ie_j^+) = -\frac{\partial}{\partial y_j} = -\sqrt{2} e_{n+j}\,,\ j = 1, \ldots, k, \tag{4.31}$$

$$\psi^{-1}(e_j^-) = \frac{\partial}{\partial x^j} = \sqrt{2} e_j\,, \quad \psi^{-1}(ie_j^-) = \frac{\partial}{\partial y_j} = \sqrt{2} e_{n+j}\,, \quad j = 1, \ldots, l\,. \tag{4.32}$$

This shows that $G^{W_o} = E_o$.

It remains to check the equivariance of $W \mapsto G^W = \psi^* g$. Using the definition of the map $\psi = \psi_W : \mathbb{R}^{2n} \to W$, one easily checks that, under the action of $\Lambda \in \mathrm{Sp}(\mathbb{R}^{2n})$, ψ transforms as

$$\psi_{\Lambda W} = \Lambda \circ \psi_W \circ \Lambda^{-1}|_{\mathbb{R}^{2n}}\,. \tag{4.33}$$

From this we deduce the transformation law of G^W:

$$G^{\Lambda W} = \psi^*_{\Lambda W} g^{\Lambda W} = (\Lambda^{-1})^* \psi^*_W \Lambda^* g^{\Lambda W} = (\Lambda^{-1})^* \psi^*_W g^W = (\Lambda^{-1})^* G^W = \Lambda \cdot G^W\,. \tag{4.34}$$

The above claim (4.27), together with the fact that

$$g^{L(p)} = g_p \quad \text{and} \quad G^{L(p)} = G(p) \tag{4.35}$$

for all $p \in \widetilde{M}$, implies that

$$\iota(L(p)) = G^{L(p)} = G(p)\,. \tag{4.36}$$

$\square$

Corollary 3 *The fundamental matrix $G : \widetilde{M} \to \mathrm{Sym}^1_{2k,2l}(\mathbb{R}^{2n})$ is pluriharmonic.*

Proof: $G = \iota \circ L$ is the composition of the holomorphic map $L : \widetilde{M} \to Gr_0^{k,l}(\mathbb{C}^{2n})$ with the totally geodesic inclusion $Gr_0^{k,l}(\mathbb{C}^{2n}) \subset \mathrm{Sym}^1_{2k,2l}(\mathbb{R}^{2n})$. The composition of a holomorphic map with a totally geodesic map is pluriharmonic. $\square$

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